Potential Nonclassical Symmetries and Solutions of Fast Diffusion Equation

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The fast diffusion equation $u_t = (u^{-1}u_x)_x$ is investigated from the symmetry point of view in development of the paper by Gandarias [Phys. Lett. A 286 (2001) 153–160]. After studying equivalence of nonclassical symmetries with respect to a transformation group, we completely classify the nonclassical symmetries of the corresponding potential equation. As a result, new wide classes of potential nonclassical symmetries of the fast diffusion equation are obtained. The set of known exact non-Lie solutions are supplemented with the similar ones. It is shown that all known non-Lie solutions of the fast diffusion equation are exhausted by ones which can be constructed in a regular way with the above potential nonclassical symmetries. Connection between classes of nonclassical and potential nonclassical symmetries of the fast diffusion equation is found.

1 Introduction

Investigation of nonlinear heat (or diffusion if u represents mass concentration) equations by means of symmetry methods was started as early as in 1959 with Ovsiannikov's work [19] where the author performed the group classification of the class of equations of the form

$$u_t = (f(u)u_x)_x. \tag{1}$$

Nonclassical symmetries of equations from class (1) were investigated in [3, 10, 11, 15]. In particular, the authors of [11] obtained the determining equations for the coefficients of conditional symmetry operators for the wider class of nonlinear reaction–diffusion equations of the form $u_t = (f(u)u_x)_x + g(u)$ and constructed a number of their exact solutions. Review of results on symmetries, exact solutions and conservation laws of such equations is given e.g. in [16].

The diffusion processes described by (1) are known to arise in different fields of physics such as plasma physics, kinetic theory of gases, solid state and transport in porous medium. In many metals and ceramic materials the diffusion coefficient $f(u)$ can, over a wide range of temperatures, be approximated as $u^{-\alpha}$, where $0 < \alpha < 2$ [29]. So, one of the mathematical model of the diffusion processes is

$$u_t = (u^{-\alpha}u_x)_x. \tag{2}$$

Equations (2) are called *fast diffusion equations* in the case $0 < \alpha < 2$ since these values of α correspond to a much faster spread of mass than in the linear case ($\alpha = 0$).

In this Letter we restrict ourselves with the special case $\alpha = 1$, i.e. with the equation

$$u_t = (u^{-1}u_x)_x. \tag{3}$$

It emerges in plasma physics as a model of the cross-field convective diffusion of plasma including mirror effects and in the central limit approximation to Calerman's model of the Boltzmann

equation. Equation (3) governs the expansion of a thermalized electron cloud described by isothermal Maxwell distribution. It is also the one-dimensional Ricci flow equation. (See [4,5,29] and references therein.)

Equation (3) has a number of remarkable mathematical properties which distinguish it from class (2). Thus, (3) can be rewritten in the form $u_t = (\ln u)_{xx}$ whereas for the other values of α the function under ∂_{xx} is a power one. It admits a discrete potential invariance transformation. For this equation wide classes of exact solutions were constructed in a closed form while reduction of (2) in the general case results in ordinary differential equations which usually cannot be integrated explicitly. Its potential form admits two kinds of variable separation.

The fact that (3) is written in a conserved form allows us, following Bluman et al. [7–9], to consider the corresponding potential system

$$v_x = u, \quad v_t = u^{-1}u_x \tag{4}$$

and to find potential symmetries of equation (3). Namely, any local symmetry of system (4) induces a symmetry of the initial equation (3). If transformations of some of the "non-potential" variables t, x and u explicitly depend on the potential v, this symmetry is a nonlocal (potential) symmetry of equation (3).

It follows from (4) that the potential v satisfies the nonlinear filtration equation

$$v_t = {v_x}^{-1}v_{xx} \tag{5}$$

with the special value ${v_x}^{-1}$ of the filtration coefficient. We will also call equation (5) the *potential fast diffusion equation*. Akhatov, Gazizov and Ibragimov carried out group classification of the nonlinear filtration equations of the general form

$$v_t = f(v_x)v_{xx} \tag{6}$$

and investigated their contact and quasi-local symmetries [1,2,16].

Lie symmetries of (3) are well known (see Section 2). All its exact solutions constructed in closed form by reduction with Lie symmetries are listed e.g. in [21].

Some non-Lie exact solutions of (3) were obtained in [15,28,29]. Thus, Rosenau [29] found that equation (5) admits, in addition to the usual variable separation $v = T(t)X(x)$, the additive one $v = Y(x+\lambda t) + Z(x-\lambda t)$ which is a potential additive variable separation for equation (3). To construct nonclassical solutions of (3), Qu [28] made use of generalized conditional symmetry method, looking for the conditional symmetry operators in the special form $Q = (u_{xx} + H(u){u_x}^2 + F(u)u_x + G(u))\partial_u$. Gandarias [15] investigated some families of usual and potential nonclassical symmetries of (2). In particular, using an ansatz for the coefficient η, she found nontrivial reduction operators in the so-called "no-go" case [12,22,30] when the coefficient of ∂_t vanishes, i.e. operators can be reduced to the form $Q = \partial_x + \eta(t,x,u)\partial_u$.

In the recent paper [9] a preliminary analysis of nonclassical symmetries of equations from class (6) was performed. A more detailed consideration was carried out for the case $f = (v_x^2 + v_x)^{-1}$, and only some examples of reduction operators and corresponding exact solutions were constructed. Let us note that equation (6) with $f = (v_x^2 + v_x)^{-1}$ is reduced by the point transformation $\tilde t = t$, $\tilde x = x + v$, $\tilde v = v$ to equation (5) which corresponds to the value $\tilde f = {\tilde v_{\tilde x}}^{-1}$ and is simpler and more convenient for investigation. All results on symmetries and exact solutions of the equation from [9] can be derived from the analogous results for equation (5).

In this Letter the fast diffusion equation (3) is investigated from the symmetry point of view. The nonclassical symmetries of the corresponding potential equation (5) are completely classified with respect to its Lie symmetry group. As a result, new wide classes of potential nonclassical

symmetries of equation (3) are found. Some classes of potential nonclassical symmetries prove to be connected with usual nonclassical ones on the solution set of potential system (4). The set of exact non-Lie solutions constructed in [15, 28, 29] is supplemented with the similar ones. It is shown that all known non-Lie solutions of the fast diffusion equation are exhausted by ones which can be constructed with the above potential nonclassical symmetries.

Our Letter is organized as follows. First of all (Section 2) we adduce results on Lie and potential symmetries of (3), including discrete ones. It is important since classical symmetries really are partial cases of nonclassical symmetries and below we solve the problem on finding only pure nonclassical symmetries which are not equivalent to classical ones. Moreover, our approach is based on application of the notion of equivalence of nonclassical symmetries with respect to a transformation group, which is developed and investigated in Section 3. Usage of equivalence with respect to the complete Lie invariance group including the discrete symmetries plays a significant role in simplification of proof, testing and improving presentation of the main result (Theorem 1, Section 4). In spite of the techniques applied in [15] and similarly to [9], we use the single potential equation (5) instead of potential system (4), to produce potential nonclassical symmetries of (3). After the "no-go" case of the zero coefficient of ∂_t is discussed, all the reduction operators having the nonvanishing coefficient of ∂_t are classified. Connection between partial classes of usual and potential reduction operators of (1) is studied in Section 5. In Section 6 the known Lie solutions of (3) and (5) are collected. A list of non-Lie solutions is supplemented with the similar ones. Connections between exact solutions and different ways of their construction are discussed shortly. In conclusion some recent results on nonclassical symmetries of equations (6) are announced.

2 Lie and potential symmetries of fast diffusion equation

The Lie invariance algebra

$$A_1 = \langle \partial_t,\ \partial_x,\ t\partial_t + u\partial_u,\ x\partial_x - 2u\partial_u \rangle$$

of equation (3) was found in [19]. The complete Lie invariance group G_1 of (3) is generated by both continuous one-parameter transformation groups with infinitesimal operators from A_1 and two involution transformations of alternating sign in the sets $\{t, u\}$ and $\{x\}$. Action of any element from G_1 on the function u is given by the formula

$$\tilde{u}(t, x) = \varepsilon_3^{-1}\varepsilon_4^2\, u(\varepsilon_3 t + \varepsilon_1, \varepsilon_4 x + \varepsilon_2),$$

where ε_1, …, ε_4 are arbitrary constants, $\varepsilon_3\varepsilon_4 \neq 0$ [19].

The Lie symmetry properties of (3) are common for diffusion equations. Uncommonness of equation (3) from the symmetry point of view becomes apparent after introducing the potential v and considering potential system (4) or potential equation (5). Point and nonclassical symmetries of (4) or (5) are called *potential* and *potential nonclassical* symmetries of (3) correspondingly.

The Lie invariance algebra

$$A_2 = \langle \partial_t,\ \partial_x,\ \partial_v,\ t\partial_t + v\partial_v,\ x\partial_x - v\partial_v \rangle$$

of equation (5) and the corresponding connected Lie symmetry group are quite ordinary for nonlinear filtration equations. However, equation (5) is distinguished for its discrete symmetries since it possesses, besides two usual sign changes in the variable sets $\{t, v\}$ and $\{x, v\}$, the hodograph transformation $\tilde{t} = t$, $\tilde{x} = v$, $\tilde{v} = x$. These three involutive transformations together with the continuous one-parameter transformation groups having infinitesimal operators

from A_2 generate the complete Lie invariance group G_2 of (5). Therefore, G_2 consists of the transformations

$$\tilde{t} = \varepsilon_3 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_4 x + \varepsilon_2, \quad \tilde{v} = \varepsilon_3 {\varepsilon_4}^{-1} v \quad \text{and}$$
$$\tilde{t} = \varepsilon_3 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_3 {\varepsilon_4}^{-1} v, \quad \tilde{v} = \varepsilon_4 x + \varepsilon_2,$$

where $\varepsilon_1, \ldots, \varepsilon_4$ are arbitrary constants, $\varepsilon_3\varepsilon_4 \neq 0$.

A similar result is true for system (4). Namely, it is invariant with respect to the following transformation

$$\tilde{t} = t, \quad \tilde{x} = v, \quad \tilde{u} = u^{-1}, \quad \tilde{v} = x \tag{7}$$

which is additional to the usual Lie symmetry group G_1 of equation (3) and is called the potential hodograph transformation of this equation.

It can be proved [25] that the set of Lie invariant solutions of equation (3) is closed under transformation (7).

3 Equivalence of reduction operators with respect to transformation groups

The notion of nonclassical symmetry was introduced in 1969 [6]. A precise and rigorous definition was suggested later (see e.g. [13, 31]).

Consider an rth order differential equation $\mathcal{L}$ of the form $L(t, x, u_{(r)}) = 0$ for the unknown function u of two independent variables t and x. Here $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to t and x of order not greater than r, including u as the derivative of order zero. Within the local approach the equation $\mathcal{L}$ is treated as an algebraic equation in the jet space $J^{(r)}$ of order r and is identified with the manifold of its solutions in $J^{(r)}$:

$$\mathcal{L} = \{(t, x, u_{(r)}) \in J^{(r)} \,|\, L(t, x, u_{(r)}) = 0\}.$$

The set of (first-order) differential operators of the general form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \quad (\tau, \xi) \neq (0, 0),$$

will be denoted by $\mathcal{Q}$. Here and below $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\partial_u = \partial/\partial u$. Subscripts of functions denote differentiation with respect to the corresponding variables.

Two differential operators $\widetilde{Q}$ and Q are called *equivalent* if they differ by a multiplier being a non-vanishing function of t, x and u: $\widetilde{Q} = \lambda Q$, where $\lambda = \lambda(t, x, u)$, $\lambda \neq 0$. The equivalence of operators will be denoted by $\widetilde{Q} \sim Q$. Denote also the result of factorization of $\mathcal{Q}$ with respect to this equivalence relation by $\mathcal{Q}_\mathrm{f}$. Elements of $\mathcal{Q}_\mathrm{f}$ will be identified with their representatives in $\mathcal{Q}$.

The first-order differential function $Q[u] := \eta(t, x, u) - \tau(t, x, u)u_t - \xi(t, x, u)u_x$ is called the *characteristic* of the operator Q. The characteristic PDE $Q[u] = 0$ (called also the *invariant surface condition*) has two functionally independent integrals $\zeta(t, x, u)$ and $\omega(t, x, u)$. Therefore, the general solution of this equation can be implicitly presented in the form $F(\zeta, \omega) = 0$, where F is an arbitrary function of its arguments.

The characteristic equations of equivalent operators have the same set of solutions. And vice versa, any family of two functionally independent functions of t, x and u is a complete set of integrals of the characteristic equation of a differential operator. Therefore, there exists a one-to-one correspondence between $\mathcal{Q}_\mathrm{f}$ and the set of families of two functionally independent functions of t, x and u, which is factorized with respect to the corresponding equivalence. (Two families

of the same number of functionally independent functions of the same arguments are considered equivalent if any function from one of the families is functionally dependent on functions from the other family.)

Since $(\tau, \xi) \neq (0,0)$ we can assume without loss of generality that $\zeta_u \neq 0$ and $F_\zeta \neq 0$ and resolve the equation $F = 0$ with respect to ζ: $\zeta = \varphi(\omega)$. This implicit representation of the function u is called an *ansatz* corresponding to the operator Q.

Denote the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u] = 0$ in $J^{(r)}$ by $\mathcal{Q}^{(r)}$, i.e.

$$\mathcal{Q}^{(r)} = \{(t, x, u_{(r)}) \in J^{(r)} \mid D_t^\alpha D_x^\beta Q[u] = 0,\ \alpha, \beta \in \mathbb{N} \cup \{0\},\ \alpha + \beta < r\},$$

where $D_t = \partial_t + u_{\alpha+1,\beta}\partial_{u_{\alpha\beta}}$ and $D_x = \partial_x + u_{\alpha,\beta+1}\partial_{u_{\alpha\beta}}$ are the operators of total differentiation with respect to the variables t and x, the variable $u_{\alpha\beta}$ of the jet space $J^{(r)}$ corresponds to the derivative $\partial^{\alpha+\beta}u/\partial t^\alpha \partial x^\beta$.

Definition 1. The differential equation $\mathcal{L}$ is called *conditionally invariant* with respect to the operator Q if the relation $Q_{(r)}L(t, x, u_{(r)})\big|_{\mathcal{L}\cap\mathcal{Q}^{(r)}} = 0$ holds, which is called the *conditional invariance criterion*. Then Q is called an operator of *conditional symmetry* (or Q-conditional symmetry, nonclassical symmetry, etc.) of the equation $\mathcal{L}$.

In Definition 1 the symbol $Q_{(r)}$ stands for the standard rth prolongation of the operator Q [18, 20]: $Q_{(r)} = Q + \sum_{0<\alpha+\beta\leqslant r} \eta^{\alpha\beta}\partial_{u_{\alpha\beta}}$, where $\eta^{\alpha\beta} = D_t^\alpha D_x^\beta Q[u] + \tau u_{\alpha+1,\beta} + \xi u_{\alpha,\beta+1}$.

The equation $\mathcal{L}$ is conditionally invariant with respect to the operator Q iff the ansatz constructed with this operator reduces $\mathcal{L}$ to an ordinary differential equation [31]. So, we will also call operators of conditional symmetry by *reduction operators* of $\mathcal{L}$.

Lemma 1 ([14, 31])**.** *If the equation $\mathcal{L}$ is conditionally invariant with respect to the operator Q, then it is conditionally invariant with respect to any operator which is equivalent to Q.*

The set of reduction operators of the equation $\mathcal{L}$ is a subset of $\mathcal{Q}$ and so will be denoted by $\mathcal{Q}(\mathcal{L})$. In view of Lemma 1, $Q \in \mathcal{Q}(\mathcal{L})$ and $\widetilde{Q} \sim Q$ imply $\widetilde{Q} \in \mathcal{Q}(\mathcal{L})$, i.e. $\mathcal{Q}(\mathcal{L})$ is closed under the equivalence relation on $\mathcal{Q}$. Therefore, factorization of $\mathcal{Q}$ with respect to this equivalence relation can be naturally restricted on $\mathcal{Q}(\mathcal{L})$ that results in the subset $\mathcal{Q}_{\rm f}(\mathcal{L})$ of $\mathcal{Q}_{\rm f}$. As in the whole set $\mathcal{Q}_{\rm f}$, we identify elements of $\mathcal{Q}_{\rm f}(\mathcal{L})$ with their representatives in $\mathcal{Q}(\mathcal{L})$. In this approach the problem of complete description of reduction operators for the equation $\mathcal{L}$ is nothing but the problem of finding $\mathcal{Q}_{\rm f}(\mathcal{L})$.

We can essentially simplify and order classification of reduction operators, additionally taking into account Lie symmetry transformations of an equation or equivalence transformations of a whole class of equations.

Lemma 2. *Any point transformation of t, x and u induces a one-to-one mapping of $\mathcal{Q}$ into itself. Namely, the transformation g: $\tilde{t} = T(t, x, u)$, $\tilde{x} = X(t, x, u)$, $\tilde{u} = U(t, x, u)$ generates the mapping $g_*\colon \mathcal{Q} \to \mathcal{Q}$ such that the operator Q is mapped to the operator $g_*Q = \tilde{\tau}\partial_{\tilde{t}} + \tilde{\xi}\partial_{\tilde{x}} + \tilde{\eta}\partial_{\tilde{u}}$, where $\tilde{\tau}(\tilde{t}, \tilde{x}, \tilde{u}) = QT(t, x, u)$, $\tilde{\xi}(\tilde{t}, \tilde{x}, \tilde{u}) = QX(t, x, u)$, $\tilde{\eta}(\tilde{t}, \tilde{x}, \tilde{u}) = QU(t, x, u)$. If $Q' \sim Q$ then $g_*Q' \sim g_*Q$. Therefore, the corresponding factorized mapping $g_{\rm f}\colon \mathcal{Q}_{\rm f} \to \mathcal{Q}_{\rm f}$ also is well-defined and one-to-one.*

Definition 2 ([23, 26])**.** The differential operators Q and $\widetilde{Q}$ are called equivalent with respect to a group G of point transformations if there exists $g \in G$ for which the operators Q and $g_*\widetilde{Q}$ are equivalent. *Notation:* $Q \sim \widetilde{Q} \bmod G$.

Lemma 3. *Given any point transformation g of the equation $\mathcal{L}$ to an equation $\tilde{\mathcal{L}}$, g_* maps $\mathcal{Q}(\mathcal{L})$ to $\mathcal{Q}(\tilde{\mathcal{L}})$ in a one-to-one manner. The same statement is true for the factorized mapping $g_{\rm f}$ from $\mathcal{Q}_{\rm f}(\mathcal{L})$ to $\mathcal{Q}_{\rm f}(\tilde{\mathcal{L}})$.*

Corollary 1. *Let G be a Lie symmetry group of the equation $\mathcal{L}$. Then the equivalence of operators with respect to the group G generates equivalence relations in $\mathcal{Q}(\mathcal{L})$ and in $\mathcal{Q}_{\rm f}(\mathcal{L})$.*

Consider the class $\mathcal{L}|_{\mathcal{S}}$ of equations $\mathcal{L}_\theta$: $L(t,x,u_{(r)},\theta(t,x,u_{(r)}))=0$ parameterized with the parameter-functions $\theta=\theta(t,x,u_{(r)})$. Here L is a fixed function of t, x, $u_{(r)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(t,x,u_{(r)})=(\theta^1(t,x,u_{(r)}),\dots,\theta^k(t,x,u_{(r)}))$ running the set $\mathcal{S}$ of solutions of the system $S(t,x,u_{(r)},\theta_{(q)}(t,x,u_{(r)}))=0$. This system consists of differential equations on θ, where t, x and $u_{(r)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order not greater than q. In what follows we call the functions θ arbitrary elements. Denote the point transformations group preserving the form of the equations from $\mathcal{L}|_{\mathcal{S}}$ by $G^\sim$.

Let P denote the set of the pairs each of which consists of an equation $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and an operator Q from $\mathcal{Q}(\mathcal{L}_\theta)$. In view of Lemma 3, action of transformations from the equivalence group $G^\sim$ on $\mathcal{L}|_{\mathcal{S}}$ and $\{\mathcal{Q}(\mathcal{L}_\theta)\,|\,\theta\in\mathcal{S}\}$ together with the pure equivalence relation of differential operators naturally generates an equivalence relation on P.

Definition 3. Let $\theta,\theta'\in\mathcal{S}$, $Q\in\mathcal{Q}(\mathcal{L}_\theta)$, $Q'\in\mathcal{Q}(\mathcal{L}_{\theta'})$. The pairs $(\mathcal{L}_\theta,Q)$ and $(\mathcal{L}_{\theta'},Q')$ are called *$G^\sim$-equivalent* if there exists $g\in G^\sim$ which transforms the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\theta'}$, and $Q'\sim g_*Q$.

Classification of reduction operators with respect to $G^\sim$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in the way that is similar to usual group classification in classes of differential equations. Namely, we construct firstly the reduction operators that are defined for all values of the arbitrary elements. Then we classify, with respect to the equivalence group, the values of arbitrary elements for each of that the equation $\mathcal{L}_\theta$ admits additional reduction operators.

In an analogues way we also can introduce equivalence relations on P, which are generated by either generalizations of usual equivalence groups or all admissible point transformations [24] (called also form-preserving ones [17]) in pairs of equations from $\mathcal{L}|_{\mathcal{S}}$.

4 Reduction operators of nonlinear filtration equation

In this section we describe G_2-inequivalent reduction operators of the potential fast diffusion equation (5). Here reduction operators have the general form $Q=\tau\partial_t+\xi\partial_x+\theta\partial_v$, where τ, ξ and θ are functions of t, x and v, and $(\tau,\xi)\neq(0,0)$. Since (5) is an evolution equation, there are two principally different cases of finding Q: $\tau\neq0$ and $\tau=0$.

In the case $\tau=0$ we have $\xi\neq0$, and up to the usual equivalence of reduction operators we can assume that $\xi=1$, i.e. $Q=\partial_x+\theta\partial_v$. The conditional invariance criterion implies only one determining equation on the coefficient θ

$$\theta\theta_t=\theta_{xx}+2\theta\theta_{xv}+\theta^2\theta_{vv}-\theta^{-1}(\theta_x)^2-2\theta_x\theta_v-\theta(\theta_v)^2$$

which is reduced with a non-point transformation to equation (5), where θ becomes a parameter. That is why the case $\tau=0$ is called the “no-go” one. It is characteristic for evolution equations in general. First the “no-go” case was completely investigated for the one-dimensional linear heat equation in [12]. It was proved that the problem of finding the conditional symmetry operators with the vanishing coefficient of ∂_t is reduced to solving the initial equation. In [30] the proof was extended to the class of $(1+1)$-dimensional evolution equations and in [22] this result was generalized for evolution equations with n space variables.

Let us note that "no-go" has to be treated as impossibility of exhaustive solving of the problem. At the same time, imposing additional constraints on the coefficient θ, one can construct a number of particular examples of operators with $\tau = 0$ and then apply them to finding exact solutions of the initial equation. It is the approach that was used in [15] for fast diffusion equation (3). Since the determining equation has more independent variables and, therefore, more freedom degrees, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. (Similar situation is for Lie symmetries of first-order ordinary differential equations.)

Consider the case $\tau \neq 0$ which admits complete solving unlike the previous case. We can assume $\tau = 1$ up to the usual equivalence of reduction operators. Then the determining equations for the coefficients ξ and θ have the form

$$\begin{aligned} &\xi_{vv} = \xi\xi_v, \qquad \xi_t = 2\xi_{xv} - \theta_{vv} - \theta_v\xi + \theta\xi_v - \xi\xi_x, \\ &\theta_{xx} = \theta\theta_x, \qquad \theta_t = 2\theta_{xv} - \xi_{xx} - \xi_x\theta + \xi\theta_x - \theta\theta_v. \end{aligned} \tag{8}$$

Theorem 1. *A complete list of G_2-inequivalent non-Lie reduction operators of the potential fast diffusion equation* (5) *is exhausted by the following ones:*

1. $\partial_t + \varepsilon\partial_x + f(\omega)\partial_v$, *where* $\omega = x + \varepsilon t$;

2. $\partial_t + f(\omega)(\partial_x + \partial_v)$, *where* $\omega = x + v$;

3. $\partial_t + \xi\partial_x + (\varphi_t + \varphi_x\xi)\partial_v$, *where* $\xi = \dfrac{-2}{v+\varphi}$, $\varphi \in \{t + e^x,\ tf(x)\}$;

4. $\partial_t + \xi\partial_x - \dfrac{\chi_t + \chi_x\xi}{1+\chi^2}\partial_v$, *where* $\xi = -2\dfrac{1+\chi\tan v}{\tan v - \chi}$,
   $\chi \in \{\tan(2t)\tanh x,\ \coth(2t)\cot x\}$;

5. $\partial_t + \xi\partial_x - \dfrac{\chi_t + \chi_x\xi}{1-\chi^2}\partial_v$, *where* $\xi = -2\dfrac{1-\chi\tanh v}{\tanh v - \chi}$,
   $\chi \in \left\{\tanh(2t)\tanh x,\ \tanh(2t)\coth x,\ \coth(2t)\coth x,\ \dfrac{e^{2x}\tanh 2t + 1}{e^{2x} - \tanh 2t},\ \dfrac{2 - e^{2x} - e^{4t}}{2 + e^{2x} + e^{4t}}\right\}$.

Here $\varepsilon \in \{0, 1\}$, f is an arbitrary nonconstant solution of the ordinary differential equation $f_{\omega\omega} = ff_\omega$, i.e. $f \in \{-2/\omega,\ -2\cot\omega,\ -2\tanh\omega,\ -2\coth\omega\} \bmod G_2$.

Proof. Here we only outline a sketch of proof. Any solution of the equation $\xi_{vv} = \xi\xi_v$ belongs to the set $\{\varphi,\ -2/(v+\varphi),\ -2\mu\cot\omega,\ -2\mu\tanh\omega,\ -2\mu\coth\omega\}$, where $\omega = \mu(v+\varphi)$, μ and φ are arbitrary functions of t and x, $\mu \neq 0$. The second equation of (8) is a linear inhomogeneous second-order ordinary differential equation with respect to θ, where v is the independent variable and t and x are assumed parameters. It is possible to construct its partial exact solution without irrational singularities for any above value of ξ. The solutions of the corresponding homogeneous equation have irrational singularities if $\xi_v \neq 0$. In view of the other equations of (8), the part of θ containing such singularities has to vanish identically. Moreover, $\mu = \text{const}$, i.e. $\mu = 1 \bmod G_2$, and φ satisfies an overdetermined system of differential equation in t and x. Integration of it for all above values of ξ and classification of obtained solutions up to equivalence with respect to G_2 with excluding Lie cases result in the statement of the theorem. $\square$

All operators from Theorem 1 are potential nonclassical symmetries of equation (3).

5 Connection between classes of nonclassical and potential nonclassical symmetries

Let us investigate connection between reduction operators of equations (3) and (5).

As mentioned in the introduction, one of the problems studied in [15] was construction of partial classes of reduction operators for diffusion equations (2) in the “no-go” case when operators can be reduced to the form $\partial_x + \eta(t,x,u)\partial_u$. Namely, Gandarias proposed to look for the coefficient η with the ansatz

$$\eta = \frac{\eta^1(t,x)u + \eta^2(t,x)}{f(u)}, \tag{9}$$

where $f(u) \equiv u^{-\alpha}$. After substituting (9) in the determining equation for η and splitting with respect to u, one obtains an overdetermined system for the functions η^1 and η^2. For the fast diffusion equation (3) this system has the form

$$\eta^2_{xx} = \eta^2\eta^2_x, \qquad \eta^2_t = \eta^2\eta^1_x - \eta^1\eta^2_x + \eta^1_{xx}, \qquad \eta^1_t = \eta^1\eta^1_x. \tag{10}$$

System (10) can be derived from (8) with reduction by the group of translations with respect to v, i.e. with assuming that ξ and θ do not depend on v and re-denoting $\xi = -\eta^1$, $\theta = \eta^2$.

This observation can be easily explained in a rigorous way for any pair of equations (1) and (6) with the same function f.

Consider reduction operators $Q = \partial_t + \xi\partial_x + \theta\partial_v$ and $Q' = \partial_x + \eta\partial_u$ of equations (6) and (1) correspondingly, where the coefficients ξ and θ depend only on t and x, the coefficient η is defined by (9). The conditional invariance criterion applied to equation (6) (or (1)) and the operator Q (Q') implies the following determining equation on ξ and θ (η^1 and η^2):

$$\begin{aligned}&(\xi\xi_x {v_x}^2 - (\xi_x\theta + \xi\theta_x)v_x + \theta\theta_x)f'(v_x)\\ &\quad +((\xi_t + 2\xi\xi_x)v_x - \theta_t - 2\theta\xi_x)f(v_x) + (-\xi_{xx}v_x + \theta_{xx})(f(v_x))^2 = 0\end{aligned}$$

$$\begin{aligned}&(\text{or} \quad (\eta^1 u + \eta^2)(\eta^1_x u + \eta^2_x)f'(u) -\\ &\quad ((\eta^1_t - 2\eta^1\eta^1_x)u + \eta^0_t - 2\eta^0\eta^1_x)f(u) + (\eta^1_{xx}u + \eta^2_{xx})(f(u))^2 = 0\ )\end{aligned}$$

which has to be additionally split with respect to v_x (u). It is obvious that the systems obtained in the both cases after splitting coincide under the supposition $\eta^1 = -\xi$, $\eta^2 = \theta$. The characteristic equation $Q[v] = \theta - v_t - \xi v_x = 0$ can be rewritten on the manifold of solutions of the potential system

$$v_x = u, \quad v_t = f(u)u_x \tag{11}$$

in the form

$$u_x - \frac{-\xi u + \theta}{f(u)} = 0$$

and coincides in this way with the characteristic equation $Q'[u] = 0$.

Therefore, the following proposition is true.

Proposition 1. *$Q = \partial_t + \xi\partial_x + \theta\partial_v$, where $\xi = \xi(t,x)$ and $\theta = \theta(t,x)$, is a reduction operator of equation (6) iff*

$$Q' = \partial_x + \frac{-\xi u + \theta}{f(u)}\partial_u$$

is a reduction operator of equation (1).

System (11) establishes connection between the corresponding sets of invariant solutions.

6 Exact solutions of fast diffusion and nonlinear filtration equations

All invariant solutions of (3) and (5), which were earlier constructed in closed forms with the classical Lie method, were collected e.g. in [21,25]. A complete list of G_1-inequivalent solutions of such type is exhausted by the following ones:

$$
\begin{aligned}
&1)\ u=\frac{1}{1+\varepsilon e^{x+t}},\quad v=-\ln|e^{-x}+\varepsilon e^{t}|;\\
&2)\ u=e^{x},\quad v=e^{x}+t;\\
&3)\ u=\frac{1}{x-t+\mu t e^{-x/t}},\quad v=\ln|t|+\int\frac{d\vartheta}{\vartheta-1+\mu e^{-\vartheta}}\bigg|_{\vartheta=x/t};\\
&4)\ u=\frac{2t}{x^2+\varepsilon t^2},\quad v\big|_{\varepsilon=0}=-\frac{2t}{x},\quad v\big|_{\varepsilon=1}=2\arctan\frac{x}{t},\quad v\big|_{\varepsilon=-1}=\ln\left|\frac{x-t}{x+t}\right|;\\
&5)\ u=\frac{2t}{\cos^2 x},\quad v=2t\tan x;\\
&6)\ u=\frac{-2t}{\cosh^2 x},\quad v=-2t\tanh x;\\
&7)\ u=\frac{2t}{\sinh^2 x},\quad v=-2t\coth x.
\end{aligned}
\tag{12}
$$

Here ε and μ are arbitrary constants, $\varepsilon\in\{-1,0,1\}$ mod G_1. The below arrows denote the possible transformations of solutions (12) to each other by means of the potential hodograph transformation (7) up to translations with respect to x [25]:

$$
\begin{aligned}
&\circlearrowright\ 1)_{\varepsilon=0}\,;\quad 1)_{\varepsilon=1}\longleftrightarrow 1)_{\varepsilon=-1,\ x+t<0}\,;\quad \circlearrowright\ 1)_{\varepsilon=-1,\ x+t>0}\,;\quad 2)\longleftrightarrow 3)_{\mu=0,\ x>t}\,;\\
&\circlearrowright\ 4)_{\varepsilon=0}\,;\quad 5)\longleftrightarrow 4)_{\varepsilon=4}\,;\quad 6)\longleftrightarrow 4)_{\varepsilon=-4,\ |x|<2|t|}\,;\quad 7)\longleftrightarrow 4)_{\varepsilon=-4,\ |x|>2|t|}\,.
\end{aligned}
$$

The sixth connection was known earlier [10,27]. If $\mu\neq 0$ solution 3) from list (12) is mapped by (7) to the solution

$$8)\ u=t\vartheta(\omega)-t+\mu t e^{-\vartheta(\omega)},\qquad \omega=x-\ln|t|,$$

which is invariant with respect to the algebra $\langle t\partial_t+\partial_x+u\partial_u\rangle$. Here ϑ is the function determined implicitly by the formula $\int(\vartheta-1+\mu e^{-\vartheta})^{-1}d\vartheta=\omega$.

Some classes of non-Lie exact solutions of (3) were obtained in [15,28,29]. These solutions and the ones similar to them can be represented uniformly over the complex field as compositions of two simple waves moving with the same "velocities" in opposite directions:

$$
\begin{aligned}
u&=\frac{\alpha^2}{\beta}(-\cot(\alpha x+\beta t+\gamma)+\cot(\alpha x-\beta t+\delta))\\
&=\frac{\alpha^2}{\beta}\frac{2\sin(2\beta t+\gamma-\delta)}{\cos(2\beta t+\gamma-\delta)-\cos(2\alpha x+\gamma+\delta)},
\end{aligned}
\tag{13}
$$

where α, β, γ and δ are complex constants, $\alpha\beta\neq 0$. It can be proved that function (13) takes real values (for real x and t) iff up to transformations from G_1

$$(\alpha,\beta,\gamma,\delta)\in\{(1,1,0,0),\ (i,i,0,0),\ (i,i,\pi/2,0),\ (i,i,\pi/2,\pi/2),\ (i,1,0,0),\ (1,i,0,0)\}.$$

Using representation (13) and the above values of tuples $(\alpha,\beta,\gamma,\delta)$, we obtain the following solutions of fast diffusion equation (3) and nonlinear filtration equation (5):

$$1')\ u=\cot(x-t)-\cot(x+t)=\frac{2\sin 2t}{\cos 2t-\cos 2x},\quad v=\ln\left|\frac{\sin(x-t)}{\sin(x+t)}\right|;$$

$$2')\ u=\coth(x-t)-\coth(x+t)=\frac{2\sinh 2t}{\cosh 2x-\cosh 2t},\quad v=\ln\left|\frac{\sinh(x-t)}{\sinh(x+t)}\right|;$$

$$3')\ u=\coth(x-t)-\tanh(x+t)=\frac{2\cosh 2t}{\sinh 2x-\sinh 2t},\quad v=\ln\left|\frac{\sinh(x-t)}{\cosh(x+t)}\right|;$$

$$4')\ u=\tanh(x-t)-\tanh(x+t)=-\frac{2\sinh 2t}{\cosh 2x+\cosh 2t},\quad v=\ln\left|\frac{\cosh(x-t)}{\cosh(x+t)}\right|;$$

$$5')\ u=\cot(ix+t)-\cot(ix-t)=\frac{2\sin 2t}{\cosh 2x-\cos 2t},\quad v=2\arctan(\cot t\,\tanh x);$$

$$6')\ u=i\cot(x+it)-i\cot(x-it)=\frac{2\sinh 2t}{\cosh 2t-\cos 2x},\quad v=2\arctan(\coth t\,\tan x).$$

(All two's in the latter expressions for u can be moved over with scale transformations from G_1.)

Transformation (7) acts on the set of solutions $1')$–$6')$ in the following way:

$$1')_{\cos 2t<\cos 2x}\longleftrightarrow 5')|_{t\to t+\pi/2,\,x\to x/2,\,v\to 2v};\quad 1')_{\cos 2t>\cos 2x}\longleftrightarrow 5')|_{x\to x/2,\,v\to 2v-\pi};$$
$$2')_{|x|<|t|}\longleftrightarrow 4')|_{x\to x/2,\,v\to 2v};\quad \circlearrowleft 2')_{|x|>|t|}|_{x\to x/2,\,v\to 2v};$$
$$\circlearrowleft 3')_{x<t}|_{x\to x/2,\,v\to 2v};\quad 3')_{x>t}\longleftrightarrow 3')_{x>t}|_{x\to -x/2,\,v\to -2v};\quad \circlearrowleft 6')|_{x\to x/2,\,v\to 2v}.$$

These actions can be interpreted in terms of actions of transformation (7) on the nonclassical symmetry operators which correspond to solutions $1')$–$6')$.

In [29] Rosenau took advantage of additive separation of variables for the potential fast diffusion equation (5) and constructed solution $4')$. Using the generalized conditional symmetry method, Qu [28] found solutions which can be written in forms $1')$ and $6')$. After rectifying computations in two cases from [28], one can find also solutions $2')$ and $5')$. Solutions $1')$, $3')$ and $4')$ were obtained in [15], at least, in one from the above forms, but equivalence of these forms was not shown there.

One of techniques which can be applied for finding the above solutions is reduction by conditional symmetry operators of the form $Q=\partial_x+(\eta^1(t,x)u+\eta^2(t,x))u\partial_u$ (see [15] for details). Namely, solutions $1')$, $3')$ and $4')$ are obtained with the following operators:

$$\partial_x+(u^2-2\cot(x-t)u)\partial_u,\quad \partial_x+(u^2-2\coth(x-t)u)\partial_u\quad\text{and}\quad \partial_x+(u^2-2\tanh(x-t)u)\partial_u.$$

We supplement the list of solutions adduced in [15, 28, 29] with similar ones, namely, with $2')$ and $5')$. Solution $2')$ can be also constructed by reduction with the second above operator. Real solutions $5')$ and $6')$ correspond to the similar operators

$$\partial_x+(iu^2-2\coth(x-it)u)\partial_u\quad\text{and}\quad \partial_x-(iu^2-2i\coth(t-ix)u)\partial_u$$

with complex-valued coefficients.

All reductions performed with reduction operators from Theorem 1 or with equivalent ones result in solutions which are equivalent to the listed Lie solutions 1)–7) or solutions $1')$–$6')$. For example, the operators

$$\partial_t-\partial_x-2\cot(x-t)\partial_v,\quad \partial_t-\partial_x-2\coth(x-t)\partial_v\quad\text{and}\quad \partial_t-\partial_x-2\tanh(x-t)\partial_v$$

lead to solutions $1')$, $3')$ and $4')$ correspondingly (see also Section 5).

7 Conclusion

In this Letter we present classification of reduction operators for nonlinear filtration equation (5) with summary of necessary notions and statements, a basic sketch of the proof and a list of constructed exact solutions including both Lie and non-Lie ones. Since (5) is the potential equation of fast diffusion equation (3), all the obtained operators are potential nonclassical symmetries of (3). Moreover, most of them are nonprojectible on the space of the independent variables t and x that leads to technically cumbersome implicit reductions of (5) to ordinary differential equations. Now we optimize the proof and hope to realize it in a quite compact form. Presentation of the proof and reduction technics will be subjects of a future paper.

We continue our investigation on potential reduction operators of the nonlinear diffusion equations from class (1). In some sense, equation (3) is singular in this class with the potential nonclassical symmetry point of view. More precisely, as a result of joint work with Prof. Sophocleous the following theorem have been proved recently.

Theorem 2. *Nonlinear filtration equations* (6) *admit non-Lie reduction operators with non-vanishing coefficients of ∂_t only in the case of the Fujita's nonlinearities*

$$f(v_x) = \frac{1}{a{v_x}^2 + bv_x + c},$$

where a, b and c are constants.

Let us note that there are exactly three $G^{\sim}$-inequivalent cases of the Fujita's nonlinearities:

$$f(v_x) = 1, \quad f(v_x) = \frac{1}{v_x}, \quad f(v_x) = \frac{1}{v_x^2+1}.$$

The equivalence group $G^{\sim}$ of class (6) is formed by the transformations

$$\tilde t = \varepsilon_1 t + \varepsilon_2, \quad \tilde x = \varepsilon_1' x + \varepsilon_2' v + \varepsilon_3', \quad \tilde v = \varepsilon_1'' x + \varepsilon_2'' v + \varepsilon_3'', \quad \tilde f = \varepsilon_1^{-1}(\varepsilon_1' + \varepsilon_2' v_x)^2 f,$$

where ε_1, ε_2, ε_i', ε_i'' ($i = 1, 2, 3$) are arbitrary constants, $\varepsilon_1(\varepsilon_1'\varepsilon_2'' - \varepsilon_2'\varepsilon_1'') \neq 0$. The nonclassical (conditional) symmetries of the (1+1)-dimensional linear heat equation ($f = 1$) were completely studied in [12]. Analogous investigation of the second case ($f = v_x^{-1}$) is carried out in this Letter. Therefore, to complete classification of reduction operators in the class of nonlinear filtration equations (6) with respect to $G^{\sim}$ (see Definition 3), it is enough to describe reduction operators of the equation with the latter nonlinearity, and we achieved significant progress in solving this problem.

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